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ASSESSING THE DEFINATION & STABILITY OF FINANCIAL EQUATION

Madhukar Dwepa
Research Scholar Monda University, Delhi Hapur Road Village & Post Kastla, Kasmabad, Pilkhuwa, Uttar Pradesh
Dr. Uma Shankar Yadav
Research Supervisor Monda University, Delhi Hapur Road Village & Post Kastla, Kasmabad, Pilkhuwa, Uttar Pradesh

ABSTRACT

The aim of this study is to the pioneering cubic functional equation and established the solution of the Ulam stability problem for these cubic mappings. It is easy to show that the function f(x) = x3 satisfies the functional equation, which is called a cubic functional equation and every solution of the cubic functional equation is said to be a cubic mapping. Several cubic functional equations and its stability investigated the stability of cubic functional equation using fixed point approach. Recently, the stability of mixed type of functional equation was discussed in the sense of Hyers, Ulam and Rassias by many authors. The stability of mixed type functional equations like additive-quadratic, quadratic-cubic, additive-quadratic-cubic some of the interesting mixed type of functional equations dealt by the different authors in various spaces. The historical background and many important results for the Ulam-Hyers stability of various functional equations are surveyed. There are applications in actuarial and financial mathematics, sociology and psychology, as well as in algebra and geometry. The study of functional equations is a contemporary area of mathematics that provides a powerful approach to working with important concepts and relationships in analysis and algebra such as symmetry, linearity and equivalence.

Keywords: - Equation, Functional, Solution, Stability.

I. INTRODUCTION

The study of functional equations is a contemporary area of mathematics that provides a powerful approach to working with important concepts and relationships in analysis and algebra such as symmetry, linearity and equivalence. Although the systematic study of such equations is a relatively recent area of mathematical study, they have been considered earlier in various forms by mathematicians such as Euler in the 18th century and Cauchy in the 19th century. The theory of functional equations is a growing branch of mathematics which has contributed a lot to the development of the strong tools in today's mathematics. Many new applied problems and theories have motivated functional equations to develop new approaches and methods. D'Alembert, Euler, Gauss, Cauchy, Abel, Weierstrass, Darboux and Hilbert are among the great mathematicians who have been concerned with functional equations and methods of solving them. Functional equations represent an alternative way of modelling problems in Physics. The interest of modelling physical problems by functional equations is that we do not have to assume the differentiability of the function f. Consequently, the functional equations lead often to other solutions than those given by partial differential equations, and these other solutions can be of interest to physicists. The most appealing characteristic of functional equation is its capacity to design mathematical models. In 1815 Charles Babbage[10] introduced a branch of mathematics now known as the theory of functional equations. But since then finding specific solutions for a given functional equation remained a hard task in many cases. For one of his examples, the now famous ”Babbage equation” $\phi(\phi(x)) = x$, which solutions ϕ are called the roots of identity and the more general equation $\phi(\phi(x)) = f(x)$, which defines a kind of a square root of some given functions f. The authors Lars Kindermann, Achim Lewandowski and Peter Protzel[90] solved this type of equations approximately by neural networks with a special topology and learning rule. Functional equations are a relatively unpopular area of mathematics. This is not due to a lack of importance: Extending linear algebra which deals with linear functions, functional algebra covers a much more general domain. Usually dynamical systems are described by differential equations in a continuous time domain. For discrete time systems on the other hand, the dynamics is defined by a difference equation or an iterated map. Constructing a trajectory or determining other properties of the system requires dealing with functional equations. Contrary to differential calculus or linear algebra, functional equations are rarely employed to solve practical problems. This may be due to the technical difficulties of the functional calculus. A 2001 survey paper on functional equations states: ”...one should not expect results on iterative roots in a general situation. In fact, even roots of polynomials are not described. Even worse: we do not know whether every complex cubic polynomial has a square root...” [20]. Facing an engineering problem which required solutions of this type of equation. The author Lars Kindermann [89] successfully developed methods to solve this equation at least numerically with the aid of neural networks. Functional equations arise in many fields of Mathematics, such as Geometry, Statistics, Probability Theory, Measure Theory, Algebraic Geometry and Group Theory. It also finds it’s applications in Information Theory, Coding Theory, Fuzzy Set Theory, Decision Theory, Game Theory, Artificial Intelligence, Cluster Analysis, Multi-Valued Logic and in many more fields. Many new applied problems and theories have motivated functional equations to develop new approaches and new methods. The theory of functional equations has been developed in a rapid and productive way in the second half of the 20th century. At the same time the self-development of this theory was very fruitful. The number of mathematical study and mathematicians dealing with functional equations is increasing day by day.

II. DEFINITION OF FUNCTIONAL EQUATIONS

A Hungarian Mathematician J. Aczel [1], an excellent specialist in functional equations, defines the functional equation as follows:

Definition 1.1.1 Functional Equation:
Functional Equations are equations in which both sides are terms constructed from the finite number of unknown functions and a finite number of independent variables.

Example
(i) f(x + y) = f(x) + f(y) (Cauchy Additive Functional Equation)
(ii) f(x + y) + f(x − y) = 2f(x) + 2f(y) (Euler-Lagrange Quadratic Functional Equation)

Definition
Ordinary Functional Equation:
A functional equation in which all the unknown functions are of one variable then it is called ordinary functional equation.
Example f(x + 2) = f(x + 1) − f(x) is an ordinary functional equation

Definition
Partial Functional Equation:
A functional equation in which at least one unknown function is a more-place function then it is called partial functional equation. Example 1.1.6 f(x + y + z) = f(x + y) + f(z) is a partial functional equation.

Definition
Rank of Functional Equation:
The number of independent variables occurring in a functional equation is called the rank of functional equation.
Example The rank of the functional equation f(x + y) = f(x) + f(y) is 2.

Definition
Solutions of Functional Equation:
A solution of a functional equation is a function which satisfies the equation.

Example
(i) Cauchy Functional Equations
f(x + y) = f(x) + f(y),
f(x + y) = f(x) f(y),
f(xy) = f(x)f(y),
f(xy) = f(x) + f(y) have solutions f(x) = kx, f(x) = e x , f(x) = cx, f(x) = lnx, respectively.
(ii) f(x) = c x + a is the solution of the Jensen functional equation

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}.$$

III. STABILITY OF FUNCTIONAL EQUATIONS

The stability of functional equations is a hot topic that has been dealt in the last five decades. In 1940, S.M. Ulam [167], gave a wide range of talk before a Mathematical Colloquium at the University of Wisconsin in which he discussed a number of important unsolved problems. One of them is the starting point of a new line of investigation, the famous Ulam Stability Problem.
Ulam Problem : Let G be a group and H be a metric group with metric d(. , .). Given > 0 does there exist a δ > 0 such that if a function f : G → H satisfies the inequality

$$d(f(xy), f(x)f(y)) < \delta$$

for all x, y ∈ G, then there exists a homomorphism a : G → H with

$$d(f(x), a(x)) < \epsilon$$

for all x ∈ G?. For the case where the answer is affirmative, the functional equation for homomorphisms will be called stable. The first result concerning the stability of functional equations was presented by D.H. Hyers [67] in 1941. He has comprehensively answered the question of Ulam for the case where G and H are Banach Spaces. He proved the following celebrated theorem.

Theorem

[67] Let X, Y be Banach spaces and let f : X → Y be a mapping satisfying

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon \qquad (1.1)$$

for all x, y ∈ X. Then the limit

$$a(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \qquad (1.2)$$

exists for all x ∈ X and a : X → Y is the unique additive mapping satisfying

$$\|f(x) - a(x)\| \leq \epsilon \qquad (1.3)$$

for all x ∈ X. Moreover, if f(tx) is continuous in t for each fixed x ∈ X, then the function a is linear. This pioneer result can be expressed as "Cauchy functional equation and it is stable for any pair of Banach space". The method which was provided by Hyers and which produces the additive function a(x) will be called a direct method. This method is the most important and most powerful tool for studying the stability of various functional equations. After publication of Hyers theorem, a large number of papers have been published in connection with various generalizations of Ulam's problem and Hyers theorem). In 1951, T. Aoki [8] generalized the Hyers theorem for approximately linear transformation in Banach spaces, by weakening the condition for the Cauchy difference for sum of powers of norms. Then Th.M. Rassias [147] in 1978, investigated a similar case (see L. Maligranda [99]). Both proved the following Hyers-Ulam-Aoki-Rassias theorem for the "sum". Th.M. Rassias [147] employed Hyers ideas to new additive mappings, and later I. Feny ([44], [45]) established the stability of the Ulam problem for quadratic and other mappings.

Theorem 1.2.2 [8, 147] Let X and Y be two Banach spaces. Let θ ∈ [0,∞) and p ∈ [0, 1). If a function f : X → Y satisfies the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \theta\left(\|x\|^p + \|y\|^p\right) \qquad (1.4)$$

for all x, y ∈ X, then there exists a unique additive mapping T : X → Y such that

$$\|f(x) - T(x)\| \leq \frac{2\theta}{2-2^p}\|x\|^p \qquad (1.5)$$

for all x ∈ X. Moreover, if f(tx) is continuous in t for each fixed x ∈ X, then the function T is linear. It is quiet remarkable to note the following Hyers' continuity condition: Suppose that f(tx) is continuous at a single point y of X, then a(x) is continuous everywhere in X. Further suppose that for each x in X the function f(tx) is a continuous function of the real variable t for −∞ < t < +∞, then a(x) is homogeneous of degree one. This condition has been assumed further till now, through the complete Hyers direct method, in order to prove linearity for generalized Hyers-Ulam stability problem forms. This result is known as the Modified Hyers - Ulam Stability for the additive functional equation. In 1982-84, J.M. Rassias [133, 134] replaced the sum by the product of powers of norms. Infact, he proved the following Ulam - Gavruta - Rassias theorem.

Theorem 1.2.3 [133, 134] Let f : E → E 0 be a mapping from a normed vector space E into a Banach space E 0 subject to the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon\|x\|^p\|y\|^p \qquad (1.6)$$

for all x, y ∈ E, where and p are constants with > 0 and 0 ≤ p < 1 2 . Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \quad (1.7)$$

exists for all x ∈ E and L : E → E 0 is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \leq \frac{\epsilon}{2 - 2^{2p}} \|x\|^{2p} \quad (1.8)$$

for all x ∈ E. If p < 0, then the inequality (1.6) holds for x, y 6= 0 and (1.8) for x 6= 0. If p > 1 2 the inequality (1.6) holds for x, y ∈ E and the limit

$$A(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) \quad (1.9)$$

exists for all x ∈ E and A : E → E 0 is the unique additive mapping which satisfies

$$\|f(x) - A(x)\| \leq \frac{\epsilon}{2^{2p} - 2} \|x\|^{2p} \quad (1.10)$$

for all x ∈ E. If in addition f : E → E 0 is a mapping such that the transformation t → f(tx) is continuous in t ∈ R for each fixed x ∈ E, then L is R− linear mapping.

IV. CONCLUSION

As a modern branch of mathematics, functional equations offer a powerful method for working with fundamental algebraic and analytic notions and relationships, such as symmetry, linearity, and equivalence. It is true that the systematic study of these equations is a relatively new area of mathematical study, but mathematicians such as Euler and Cauchy have examined them earlier in various forms. In today's mathematics, the theory of functional equations has played a significant role in the development of many powerful tools. Functional equations have been inspired by a wide range of new practical problems and theories. Among the notable mathematicians who have studied functional equations and methods for solving them are D'Alembert, Euler, Gauss, Cauchy, Abel, Weierstrass, Darboux, and Hilbert. In physics, functional equations can be used as an alternate method of describing a problem. The benefit of using functional equations to model physical problems is that we do not have to assume that function f is differentiable. Thus, functional equations often provide alternative solutions than partial differential equations, and these additional solutions may be of interest to physicists. Most interesting about functional equation is its ability to create mathematical models. Babbage[10] established the theory of functional equations in 1815, a field of mathematics currently known as functional equation theory. Finding specific solutions to a given functional equation has remained difficult in many circumstances since that time. He used the now-famous "Babbage equation" (x) = x as one of his examples. We can refer to these values as the roots of identity, and they can also be found in the more general equation, f(x), where they represent the square root of some function f. Lars Kindermann, Achim Lewandowski, and Peter Protzel[90] used neural networks with a particular topology and learning method to approximate this type of equation. The study of functional equations is not widely practised. Because this isn't a low priority issue: The scope of functional algebra extends well beyond that of linear algebra's focus on linear functions. Differenz equations are commonly used to characterise dynamical systems in a continuous time domain. A difference equation or iterated map characterises the dynamics of discrete temporal systems, in contrast. Functional equations are necessary for constructing a trajectory or establishing the system's other attributes. Functional equations, unlike differential calculus or linear algebra, are rarely used in practise. This could be because of the functional calculus's technical problems. "...one should not expect results on iterative roots in a general context," states a 2001 survey report on functional equations. In reality, nothing about polynomial roots is explained. We don't even know if every complex cubic polynomial has a square root. [20]. This type of equation had to be solved for in order to solve an engineering challenge. The author Lars Kindermann [89] was able to use neural networks to build methods for solving this equation quantitatively at the very least. Functional equations can be found in a wide range of mathematical disciplines, such as geometry, algebraic geometry, probability theory, statistics, and measure theory. Fuzzy set theory, game theory, decision-theory, decision-theory cluster analysis and multi-valued logic are just a few of the many domains in which it has been used successfully. Functional equations have been inspired by a wide range of new applied problems and ideas to produce new approaches and methods. In the second part of the twentieth century, the theory of functional equations was rapidly and productively developed. Meanwhile, this theory's growth was extremely fruitful. Functional equations are being studied and studied by an increasing number of mathematicians.

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