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A STUDY OF PARTIAL FUNCTIONS AND COMPOSITION OF MATHEMATICAL FORMULATION ON ALGEBRAS

CANDIDATE NAME- AKHILESH KUMAR YADAV
DESIGNATION- RESEARCH SCHOLAR SUNRISE UNIVERSITY ALWAR
GUIDE NAME - DR. DIPITI VASDEV
DESIGNATION- Professor SUNRISE UNIVERSITY ALWAR

ABSTRACT

This study examines numerical approximation of across mathematics as a whole; functions have a much more pervasive presence than relations, their more general cousins. However, with regard to the metamathematics of reasoning about these entities, the situation has, historically, been reversed. Initially, the purpose of this work was algebraic logic in the strict sense. That is to say, the relations were providing the semantics for *logical formulas*. This is perhaps the explanation as to why, until recently, the corresponding theory of functions was relatively less developed, since the semantics of formulas is not a role so naturally suited to functions. However, increasingly in the history of relations, computer science has become a source of motivation, with binary relations providing the semantics for, in particular, *(nondeterministic) computer programs*. In this view, the relation relates states of the machine before the program is executed to possible states after it is executed.

KEYWORDS: Mathematical Formulation, Algebras, Partial Functions, numerical approximation, logical formulas

INTRODUCTION

Despite the ubiquity of functions in mathematics, prior to the turn of this century the only sustained period of activity on reasoning with (possibly partial) functions was the 1960s, when the semi group theorist Boris Schein and associates were active in this area.[2,3] In the last fifteen years however, interest has rekindled, and a regular stream of papers has been appearing, with computer science considerations a prime motivation. Whenever we have a concrete class of algebras whose operations are set-theoretically defined, we have a notion of a representation: an isomorphism from an abstract algebra to a concrete algebra. Then the representation class—the class of representable algebras—becomes an object of interest itself. One possibility—the focus of this thesis—is for the concrete algebras to be algebras of partial functions, and for this scenario various signatures have been considered. Often, the representation classes have turned out to be finitely axiomatisable varieties or quasivarieties.

Algebras of partial functions

Various classes of algebras of relations Algebras of partial functions are algebras of functional relations, which for a binary relation R means

$$xRy \wedge xRy' \rightarrow y = y'$$

for all x, y and y'. Hence algebras of partial functions are simply yet further variants of algebras of relations, and the methodology we use is exactly the same: each choice of set-theoretic operations gives a notion of representability for abstract algebras, and we can then study the representation class and

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related issues such as finite or complete represent ability.

Unary functions

The basic and most common case is to consider **unary partial functions**, that is, functional binary relations on some base set *X*.

We first give a non-exhaustive list of operations that have appeared in work on algebras of unary partial functions. So great a proportion of the operations have only a single symbol for both set-theoretic and the abstract operations that in this section we only use one symbol set. We have already met

- function composition: ; (a special case of relation composition),[8]
- intersection: ·
- empty function: 0
- identity function: 1’ (defined on the specified base), and there is also
- **domain**: D a unary operation—D(*f*) is the identity function restricted to the domain of *f* ,
- **antidomain**: A a unary operation—A(*f*) is the identity function restricted to those points in the base where *f* is *not* defined,
- **range**: R a unary operation—R(*f*) is the identity function restricted to the range of *f* ,
- **fixset**: F a unary operation—F(*f*) is the identity function restricted to the fixed points of *f* ,
- **preferential union**: H a binary operation—the preferential union of *f* and *g* takes the value of *f* where *f* is defined and the value of *g* where *f* is not defined and *g* is,
- **relative complement**: \ the usual binary relative complement operation on sets,
- **maximum iterate**: ↑ a unary operation—$f^{\uparrow}(x)$ is defined if only a finite number of iterations of *f* are defined on *x* and takes the value $f^n(x)$ for the maximum value of *n* that this is defined. So

$$f^{\uparrow}(x) = \bigcup_{n \in \mathbb{N}} (f^n \,;\, \mathrm{A}(f)).$$

The reader will note there are operations featuring heavily in the section on binary relations but absent in the above list. If an operation on partial functions does not *in general* yield a function, then it is not terribly useful to be able to reason about algebras of partial functions with that operation in the signature. Firstly, if we ever want to apply a validity of such algebras to any specific functions, we are burdened with proving that those functions can coexist in an algebra of that signature and not generate a non-function. Secondly, such algebras are often so restricted as to not be interesting. Take collections of partial functions closed under unions: there cannot be even one point that can map to more than one place. Signatures containing complement are even worse: there could not be more than two points in the base.

The restriction to a single fixed base set *X* is for many signatures not important, as we can reduce to the single base case by taking a union of bases. This is not true of signatures containing antidomain though, because the antidomain operation is corrupted by expanding the base. Nevertheless, throughout this thesis we only concern ourselves with the single-base-set setup.

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Having indicated a correspondence between operations on partial functions and symbols, as we have done above, we get a definition of representation by partial functions for any signature containing any combination of symbols in the correspondence. Let Σ be such a signature and A a Σ-algebra. A **representation of** A **by partial functions** is an isomorphism from A to an algebra whose elements are partial functions and whose interpretations are the indicated operations.

COMPLETE REPRESENTATION BY PARTIAL FUNCTIONS FOR COMPOSITION, INTERSECTION AND ANTIDOMAIN

Extra conditions we can impose on a representation are to require that it be meet complete or to require that it be join complete. A representation is meet complete if it turns any existing infima into intersections and join complete if it turns any existing suprema into unions. Hence we can define meet- complete representation classes and join-complete representation classes. In many important cases these two classes coincide. Bounded distributive lattices represented as rings of sets are an example where they do not.

In, Hirsch and Hodkinson showed that when the representation class is elementary, the com- plete representation class may (as is the case for Boolean algebras represented as fields of sets) or may not (relation algebras by binary relations) also be elementary.

In this study we investigate complete representation by partial functions for the signature {;, ·, A} of composition, intersection, and antidomain. We see that for this particular signature the algebras behave in many ways like Boolean algebras. We show that, as one consequence of this similarity to Boolean algebras, a representation by partial functions is meet complete if and only if it is join complete.

We show that a representation is complete if and only if it is atomic. We use the requirement that completely representable algebras be atomic to prove that the class of completely rep- resentable algebras is not closed under subalgebras, directed unions or homomorphic images and is not axiomatisable by any existential-universal-existential first-order theory.

We investigate the validity of various distributive laws with respect to the classes of representable and completely representable {;, ·, A}-algebras. This enables us to give an example of an algebra that is representable and atomic, but not completely representable.

We present an explicit representation, which we use, to prove our main result: the class of completely representable algebras is a basic elementary class, axiomatisable by a universal-existential-universal first-order sentence.

Representations and complete representations

In this section we give preliminary definitions and then proceed to show that for the signature {; ·, A}, a representation by partial functions is meet complete if and only if it is join complete.

Given an algebra A, when we write $a \in$ A or say that a is an element of A, we mean that a is an element of the domain of A. Similarly for the notation $S \subseteq$ A or saying that S is a

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subset of A. The notation |A| denotes the cardinality of the domain of A. We follow the convention that algebras are always nonempty. If S is a subset of the domain of a map θ then $\theta[S]$ denotes the set $\{\theta(s) \mid s \in S\}$. If S_1 and S_2 are subsets of the domain of a binary operation $*$ then $S_1 * S_2$ denotes the set $\{s_1 * s_2 \mid s_1 \in S_1 \text{ and } s_2 \in S_2\}$. In a poset P (whose identity should be clear) the notation $\downarrow a$ signifies the down-set
$\{b \in P \mid b \leq a\}$.

Definition 1. Let σ be an algebraic signature whose symbols are a subset of {;, ·, 0, 1', D, R, A}. Algebra **of partial functions** of the signature σ is an algebra of the signature σ whose elements are partial functions and with operations given by the set-theoretic operations on those partial functions described in the following.

Let X be the union of the domains and ranges of all the partial functions. We call X the **base**. In an algebra of partial functions

- the binary operation ; is composition of partial functions:

$f ; g = \{(x, z) \in X^2 \mid \exists y \in X : (x, y) \in f \text{ and } (y, z) \in g\}$,

- the binary operation · is intersection:

$f \cdot g = \{(x, y) \in X^2 \mid (x, y) \in f \text{ and } (x, y) \in g\}$,

- the constant 0 is the nowhere-defined function:

$0 = \emptyset$,

- the constant 1' is the identity function on X:

$1' = \{(x, x) \in X^2\}$,

- the unary operation D is the operation of taking the diagonal of the domain of a function:

$D(f) = \{(x, x) \in X^2 \mid \exists y \in X : (x, y) \in f\}$,

- the unary operation R is the operation of taking the diagonal of the range of a function:

$R(f) = \{(y, y) \in X^2 \mid \exists x \in X : (x, y) \in f\}$,

- the unary operation A is the operation of taking the diagonal of the antidomain of a func tion—those points of X where the function is not defined:

$A(f) = \{(x, x) \in X^2 \mid \nexists y \in X : (x, y) \in f\}$.

The list of operations in Definition 1 does not exhaust those that have been considered for partial functions but does include the most commonly appearing operations.

Definition 2. Let A be an algebra of one of the signatures specified by Definition 1. A **repres- entation of** A **by partial functions** is an isomorphism from A to algebra of partial functions of the same signature. If A has a representation then we say it is **representable**.

Theorem 1 (Jackson and Stokes)**.** *The class of* {;, ·, A}*-algebras representable by partial func- tions is a finitely based variety.*

In fact in finite equation axiomatisation of the representation class is given, implicitly. So there exist known examples of such axiomatisations.

If an algebra of the signature {;, ·, A} is representable by partial functions, then it forms a ·- semilattice. Whenever we treat such an algebra as a poset, we are using the order induced by this semilattice.

The next two definitions apply to any situation where the concept of a representation has been defined. So in

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particular, these definitions apply to representations as fields of sets as well as to repres- entations by partial functions.

Definition 3. A representation θ of a poset P over the base X is **meet complete** if, for every nonempty subset S of $\mathfrak{P}$, if $\prod S$ exists, then

$$\theta(\prod S) = \bigcap \theta[S].$$

Definition 4. A representation θ of a poset P over the base X is **join complete** if, for every subset S of $\mathfrak{P}$, if $\sum S$ exists, then

$$\theta(\sum S) = \bigcup \theta[S].$$

Note that S is required to be nonempty in Definition 4 but not in Definition 5. For rep- resentations of Boolean algebras as fields of sets, the notions of meet complete and join complete are equivalent, so in this case we may simply use the adjective **complete**.

Note that if A is an algebra of the signature {;, ·, A} and A is representable by partial functions, then A must have a least element, 0, given by A(a) ; a for any $a \in$ A and any representation must represent 0 with the empty set. Similarly D := A^2 must be represented by the set-theoretic domain operation.

The following lemma demonstrates the utility of the particular signature {;, ·, A}. The similarity of representable {;, ·, A}-algebras to Boolean algebras allows results from the theory of Boolean algebras to be imported into the setting of {;, ·, A}-algebras.

Lemma 1. Let A be an algebra of the signature {;, ·, A}. If A is representable by partial functions, then for every a ∈ A, the set ↓ a, with least element 0, greatest element a, meet given by ·, and comple- mentation given by b := A(b) ; a is a Boolean algebra. Any representation θ of A by partial functions restricts to a representation of ↓ a as a field of sets over θ(a). If θ is a meet-complete or join-complete representation, then the representation of ↓ a is complete.

Proof. If θ is a representation of A by partial functions, then $b \leq a \implies \theta(b) \subseteq \theta(a)$, so θ does indeed map elements of $\downarrow a$ to subsets of $\theta(a)$. We have $b, c \in \downarrow a \implies b \cdot c \in \downarrow a$ and $\theta(b \cdot c) = \theta(b) \cap \theta(c)$ is always true by the definition of functional representability. For $b \leq a$

$\theta(b) = \theta(\mathrm{A}(b) ; a) = \mathrm{A}(\theta(b)) ; \theta(a) = \theta(a) \setminus \theta(b)$,

so $b \in \downarrow a$ and $\theta(b) = \theta(b)^c$, where the set complement is taken relative to $\theta(a)$. Hence the restriction of θ to $\downarrow a$ is a representation of ($\downarrow a$, 0, a, ·,) as a field of sets over $\theta(a)$ (from which it follows that $\downarrow a$ is a Boolean algebra).

THE FINITE REPRESENTATION PROPERTY FOR COMPOSITION, INTERSECTION, DOMAIN, AND RANGE

The investigation of the abstract algebraic properties of partial functions involves studying the isomorph- ism class of algebras whose elements are partial functions and whose operations are some specified set of operations on partial functions—operations such as composition or intersection, for example. We refer to algebra isomorphic to an algebra of partial functions as representable.

As we have indicated in previous chapters, one of the primary aims is to determine how simply the class of representable algebras can be axiomatised and to find such an axiomatisation. Often, the representation classes have turned out to be axiomatisable by finitely many equations or quasi-equations ; we detailed this earlier,

Another question to ask is whether every finite representable algebra can be represented by partial functions on some finite set. Interest in this so-called finite representation property originates from its potential to help prove decidability of representability, which in turn can

Recently, Hirsch, Jackson, and Mikula´s established the finite representation property for many sig- natures, but they leave the case for signatures containing the intersection, domain, and range operations together open.

In this chapter we prove the finite representation property for the most significant group of outstanding signatures, which includes a signature containing all the most commonly considered operations on partial functions. From our proof we obtain a double-exponential bound on the size of base set required for a representation. It follows as a corollary that representability of finite algebras is decidable for all these signatures. As an additional observation, we give an example showing that there are signatures for which the finite representation property does not hold for representation by partial functions.

The results presented here originate with McLean. The contribution of the second author is to translate the original proof of the finite representation property into a semantical setting, so that the presence of antidomain is not necessary.

Algebras of partial functions

In this section we give the fundamental definitions that are needed in order to state the results contained in this chapter.

Given an algebra A, when we write $a \in$ A or say that a is an element of A, we mean that a is an element of the domain of A. We follow the convention that algebras are always nonempty.

Definition 1. Let σ be an algebraic signature whose symbols are a subset of {;, ·, D, R, 0, 1', A, F, ∐,↑, $^{-1}$}. An **algebra of partial functions** of the signature σ is an algebra of the signature σ whose elements are partial functions and with operations given by the set-theoretic operations on those partial functions described in the following.

Let X be the union of the domains and ranges of all the partial functions occurring in an algebra ϗ. We call X the **base** of A. The interpretations of the operations in σ are given as follows:

- the binary operation ; is **composition** of partial functions:

$f ; g = \{(x, z) \in X^2 \mid \exists y \in X : (x, y) \in f \text{ and } (y, z) \in g\}$,

that is, $(f ; g)(x) = g(f(x))$,

- the binary operation · is **intersection**:

$f \cdot g = \{(x, y) \in X^2 \mid (x, y) \in f \text{ and } (x, y) \in g\}$,

- the unary operation D is the operation of taking the diagonal of the **domain** of a function:

$D(f) = \{(x, x) \in X^2 \mid \exists y \in X : (x, y) \in f\}$,

- the unary operation R is the operation of taking the diagonal of the **range** of a function:

$R(f) = \{(y, y) \in X^2 \mid \exists x \in X : (x, y) \in f\}$,

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- the constant 0 is the nowhere-defined **empty function**:

$0 = \emptyset$,

- the constant 1' is the **identity function** on X:

$1' = \{(x, x) \in X^2\}$,

- the unary operation A is the operation of taking the diagonal of the **antidomain** of a func- tion—those points of X where the function is not defined:

$A(f) = \{(x, x) \in X^2 \mid \exists / y \in X : (x, y) \in f\}$,

- the unary operation F is **fixset**, the operation of taking the diagonal of the fixed points of a function:

$F(f) = \{(x, x) \in X^2 \mid (x, x) \in f\}$,

- the binary operation ⊔ is **preferential union**:

$$(f \sqcup g)(x) = \begin{cases} f(x) & \text{if } f(x) \text{ defined} \\ g(x) & \text{if } f(x) \text{ undefined, but } g(x) \text{ defined} \\ \text{undefined} & \text{otherwise} \end{cases}$$

- the unary operation $^\uparrow$ is the **maximum iterate**:

$$f^\uparrow = \bigcup_{n \in \mathbb{N}} (f^n \,;\, A(f)),$$

where $f^0 := 1'$ and $f^{n+1} := f \,;\, f^n$,

- the unary operation $^{-1}$ is an operation we call **opposite**:

$f^{-1} = \{(y, x) \in X^2 \mid (x, y) \in f \text{ and } ((x', y) \in f \Longrightarrow x = x')\}$.

The list of operations in Definition 1 does not exhaust those that have been considered for partial functions but does include the most commonly appearing operations.

Definition 2. Let A be an algebra of one of the signatures permitted by Definition 1. A **repres- entation of** A **by partial functions** is an isomorphism from ϗ to an algebra of partial functions of the same signature. If A has a representation then we say it is **representable**.

In Jackson and Stokes give a finite equational axiomatisation of the representation class for the signature {;, ·, D, R} and similarly for any expansion of this signature by operations in {0, 1', F}.

Hirsch, Jackson, and Mikula´s give a finite equational axiomatisation of the representation class for the signature {;, ·, A, R} and similarly for any expansion of this signature by operations in {0, 1', D, F, H}. For expanded signatures containing the maximum iterate operation they give finite sets of axioms that, if we restrict attention to finite algebras, axiomatise the representable ones.

The operation that we call opposite is described in where Menger calls the concrete operation 'bilateral inverse' and uses 'opposite' to refer to an abstract operation intended to model this bilateral inverse. The opposite operation appears again in Schweizer and Sklar's _and but thereafter does not appear to have received any further attention. In particular, for signatures containing opposite, axiomatisations of the representation classes remain to be found.

CONCLUSION

We have added to the signatures for which the finite representation property is known, including a signature expressing almost every operation that has been considered.

For multiplace functions, we have added finite axiomatisability results and results on the complexity of equational theories, and we have begun the investigation of complete representability for partial functions, obtaining a finite axiomatisability result. This again contrasts with relations. These are all positive results; we obtained some more finite axiomatisations, but also showed some representation classes are not finitely axiomatisable. It would be interesting to investigate exactly what causes this divergence from our other results and all those that have come before. For reasoning with partial functions, the application we gave the most prominence to was using functions to model the dynamic action of computer programs. So it is worth discussing what might be necessary to reason in a way that has practical value, and whether this is feasible. For applications, it is deciding the validity of formulas, more than having axiomatisations or deciding representability that is useful. Of course, the fewer the syntactic restrictions on the formulas under consideration, the more complex deciding validity is likely to become. Reducing this to proving an equation, the task cannot be fully automated, for the equation is only valid on condition of the validity of two simpler equations, which then have to be verified by hand after instantiating variables to atomic programming statements. Viewed another way, if the right relationships between atomic statements are known and supplied to the automated prover, and then being able to deduce a quasiequational validity is precisely what is needed for the prover to perform the verification task.

REFERENCES

1. Karapetyants, A., Samko, S. Variable order fractional integrals in variable generalized Hölder spaces of holomorphic functions. Anal.Math.Phys. 11, 156 (2021). https://doi.org/10.1007/s13324-021-00587-0
2. Agirseven, D. “Approximate solutions of delay parabolic equations with the Dirichlet condition”, Abstr. Appl. Anal., Vol. 2012, Article ID 682752, 2012.
3. Anh, C.T. and Hieu, L.V. “Existence and uniform asymptotic stability for an abstract differential equation with infinite delay”, Electron J. Diff. Equ., Vol. 51, pp. 1–14, 2012.
4. Chang, J.C. “Existence of some neutral partial differential equations with infinite delay”, Nonlinear Anal., Vol. 74, No. 10, pp. 3204–3217, 2011.
5. Condon, M., Deano, A., Iserles, A. and Kropielnicka, K. “Efficient computation of delay differential equations with highly oscillatory terms”, Math. Model. Numer. Anal., Vol. 46, No. 6, pp. 1407–1420, 2012.
6. Gokdogan, A., Merdan, M. and Yildirim, A. “Differential transformation method for solving a neutral functional-differential equation with proportional delays”, Caspian J. Math. Sci., Vol. 1, No. 1, pp. 31–37, 2012.

7. Huang, C. and Vandewalle, S. "Unconditionally stable difference methods for delay partial differential equations", Numer. Math., Vol. 122, pp. 579–603, 2012.
8. Jian, M., Zheng, B. and Zhang, C. "A matrix method for determining eigen values and stability of singular neutral delay-differential systems", J. Appl. Math., Vol. 2012, Article ID. 749847, 2012.
9. Krol, K. "Asymptotic properties of fractional delay differential equations", Appl. Math. Comput., Vol. 218, pp. 1515–1532, 2011.
10. Liu, X., Zhong, S. and Ding, X. "Robust exponential stability of impulsive switched systems with switching delays: A Razumikhin approach", Commun. Nonlinear Sci. Numer. Simul., Vol. 17, No. 4, pp. 1805–1812, 2012.
11. Roales, M., Rodrguez, F. and Martin, J.A. "Exact solutions and numerical approximations of mixed problems for the wave equation with delay", Appl. Math. Comput., Vol. 219, pp. 3178–3186, 2012.
12. Siddiqu, S.S., Akram, G. and Baig, H. "Solution of delay differential equations using nonic spline collocation methods", Int. Math. Forum, Vol. 7, No. 46, pp. 2279–2292, 2012.